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HW5

MATH 527B
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1. Exercises 4.3.31 and 4.3.32, Pg I-225 in the class notes. What is the relation between this result and the Riemann-Lebesgue lemma?

   (4.3.31) Show that the rule

   $$\varphi \mapsto \int_{-\infty}^{\infty} e^{ixt}\varphi(x)dx$$

   defines a distribution on $C_0^\infty(\mathbb{R})$.

   Solution. To show that T is a distribution, it must be both linear and continuous. Linearity is trivial (look at the definition of the integral operator) and are left with simply showing that T is continuous. We need that

   $$\int_{-\infty}^{\infty} e^{ixt}\phi_m(x)dx \to \int_{-\infty}^{\infty} e^{ixt}\phi(x)dx$$

   whenever $\phi_m \to \phi$. Making use of the linearity of T then, I will equivalently show that

   $$\int_{-\infty}^{\infty} e^{ixt}\left(\phi_m(x) - \phi(x)\right)dx \to 0$$

   Now, given that $\phi_m, \phi \in C_0^\infty(\mathbb{R})$, then we will let the support of ϕ be bounded, that is $\exists R < \infty$ such that $\text{supp}(\phi) = \{x : |x| < R\}$. We can then bound the difference of the test function

   $$|\phi_m(x) - \phi(x)| \le p_{R,0}(\phi_m - \phi)$$

   and we also have that $\sup_{x<R}|e^{ixt}| \le C|e^{it}| \le C < \infty$. So

   $$\begin{aligned}
   \left|\int_{-\infty}^{\infty} e^{ixt}\left(\phi_m(x) - \phi(x)\right)dx\right| &\le \int_{-\infty}^{\infty} \left|e^{ixt}\right| \left|\phi_m(x) - \phi(x)\right| dx \\
   &\le \int_{-R}^{R} Cp_{R,0}(\phi_m - \phi)dx \\
   &= 2RCp_{R,0}(\phi_m - \phi) \\
   &\Downarrow \\
   \lim_{m\to\infty}\left|\int_{-\infty}^{\infty} e^{ixt}\left(\phi_m(x) - \phi(x)\right)dx\right| &= 0
   \end{aligned}$$

   Therefore since T is a linear and continuous functional, we have that it is a distribution on $C_0^\infty(\mathbb{R})$.

(4.3.32) Continue the preceding exercise. Let T_j be the distribution defined by $\exp ijx$ Show that $T_j \to 0$ in the sense of Definition 4.3.20. (Hint: write

$$\int_{-\infty}^{\infty} e^{ijx}\varphi(x)dx = \frac{1}{ij}\int_{-\infty}^{\infty}\frac{d}{dx}(e^{ijx})\varphi(x)dx$$

and integrate by parts. See Example 3.7.18).

Solution.

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{ijx}\phi(x)dx &= \frac{1}{ij}\int_{-\infty}^{\infty}\frac{d}{dx}(e^{ijx})\phi(x)dx \\
&= \frac{1}{ij}\left[e^{ijx}\phi(x)\big|_{\mathbb{R}} - \int_{\mathbb{R}} e^{ijx}\phi'(x)dx\right] \\
&= -\frac{1}{ij}\int_{\mathbb{R}} e^{ijx}\phi'(x)dx, \text{ since } \phi \in C_0^\infty(\mathbb{R})
\end{aligned}$$

Now take the limit,

$$\begin{aligned}
\lim_{j\to\infty}\left|-\frac{1}{ij}\int_{\mathbb{R}} e^{ijx}\phi'(x)dx\right| &\leq \lim_{j\to\infty}\frac{1}{j}\int_{\mathbb{R}}\left|e^{ijx}\right||\phi'(x)|dx \\
&\leq \lim_{j\to\infty}\frac{C}{j}\int_{-R}^{R}\frac{1}{1+x^2}(1+x^2)|\phi(x)|dx \\
&\leq \lim_{j\to\infty}\frac{C}{j}\sup_{|x|<R}|(1+x^2)\phi'(x)|\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx \\
&= \lim_{j\to\infty}\frac{C\pi}{j}\sup_{|x|<R}|(1+x^2)\phi'(x)| \to 0
\end{aligned}$$

since $\sup_{|x|<R}|(1+x^2)\phi'(x)| < \infty$ since ϕ is a compactly supported, continuously differentiable function, we have that

$$\lim_{j\to\infty}\left|-\frac{1}{ij}\int_{\mathbb{R}} e^{ijx}\phi'(x)dx\right| = 0$$

Therefore the distribution T_j converges to 0, in the sense of Definition 4.3.20.

♣

The Riemann-Lebesgue lemma states that, if $f \in L^1(\mathbb{R})$, then

$$\lim_{|k|\to\infty}\hat{f}(k) = 0.$$

The connection seems almost transparent, as the distribution from the previous exercise, call it T_k now for commonality, converges to 0 in the sense of Definition 4.3.20. The distribution T_k is the same in definition as the Fourier transform of an L^1 function, since $\phi \in L^1$ (due to its compact support).

2. Exercise 4.3.34, Pg. I-225.

   (4.3.34) Show that the sum

$$\sum_{k=-\infty}^{\infty} a_k \delta(x-k) \tag{1}$$

   converges in $\mathcal{D}'(\mathbb{R})$ (see 4.3.20)) for arbitrary $a_k \in \mathbb{C}$. [Hint: convergence of the series means: the partial sums $\sum_{-N}^{M}$ converge in the sense of Definition (4.3.20) as $N \to \infty$, $M \to \infty$.]

   Solution. Given that $\phi \in C_0^\infty$, then there exists some $M \in \mathbb{R}$ such that $\text{supp}(\phi) \in [-M, M]$. Then

$$\begin{aligned} \sum_{-\infty}^{\infty} a_k \delta(x-k)[\phi] &= \sum_{-\infty}^{\infty} a_k \phi(k), \text{ but since } \phi \in C_0^\infty \\ &= \sum_{-M}^{M} a_k \phi(k) \end{aligned}$$

   where we use the fact that for all $k \geq M$, $k \notin \text{supp}(\phi)$. Hence for all such k,

$$\sum_{-\infty}^{\infty} a_k \phi(k) = \sum_{-M}^{M} a_k \phi(k)$$

   and so we have that every partial sum will converge, which implies that Eqn 1 also converges.

3. **Greens function for Poissons equation**

   Let χ denote the space of smooth functions whose support is contained in $(0,1)$, with $f_n \to f$ in χ, if there is a closed set $K \subset (0,1)$ such that the supports of f_n and f are all contained in K, and the functions f_n along with all of their derivatives converge uniformly on K. Let χ' denote the distributions on χ.

   (a) $y \in (0,1)$. Find a function $G_n(x,y)$ that solves

$$\partial_x^2 G_n(x,y) = \begin{cases} \frac{n}{2} & |x-y| \leq \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$

   with the boundary conditions $G_n(0,y) = G_n(1,y) = 0$.

   Solution. Integrating the above equation once, we have, setting $G_y(x) = u(x)$

$$
\begin{aligned}
u_x(x) - u_x(y - \tfrac{1}{n}) &= \int_{y-\frac{1}{n}}^{x} \frac{n}{2} dx \\
u_x(x) &= \int_{y-\frac{1}{n}}^{x} \frac{n}{2} dx + u_x(y - \tfrac{1}{n}), \text{ integrating again} \\
u(x) &= \int_{y-\frac{1}{n}}^{x} \int_{y-\frac{1}{n}}^{x} \frac{n}{2} dx_2 dx_1 + x u_x(y - \tfrac{1}{n}) + u(y - \tfrac{1}{n}) \\
&= \int_{y-\frac{1}{n}}^{x} \frac{n}{2}(x - \xi) d\xi + x u_x(y - \tfrac{1}{n}) + u(y - \tfrac{1}{n})
\end{aligned}
$$

Imposing the boundary condition $u(0) = 0$, then gives us

$$
\begin{aligned}
0 &= \int_{y-\frac{1}{n}}^{0} -\frac{n}{2} \xi d\xi + 0 + u(y - \tfrac{1}{n}), \text{ thus} \\
u(y - \tfrac{1}{n}) &= \int_{y-\frac{1}{n}}^{0} \frac{n}{2} \xi d\xi
\end{aligned}
$$

Now using the other boundary condition, that $u(1) = 0$

$$
\begin{aligned}
0 &= \int_{y-\frac{1}{n}}^{y+\frac{1}{n}} \frac{n}{2}(1 - \xi) d\xi + u_x(y - \tfrac{1}{n}) + u(y - \tfrac{1}{n}), \text{ and so} \\
u_x(y - \tfrac{1}{n}) &= \int_{-1}^{1} \frac{n}{2}(\xi - 1) d\xi - u(y - \tfrac{1}{n}) \\
&\Downarrow \\
u(x) &= \int_{y-\frac{1}{n}}^{x} \frac{n}{2}(x - \xi) d\xi + x \int_{y-\frac{1}{n}}^{y+\frac{1}{n}} \frac{n}{2}(\xi - 1) d\xi \\
&= \int_{y-\frac{1}{n}}^{y+\frac{1}{n}} k(x, \xi) d\xi
\end{aligned}
$$

where

$$
k(x, \xi) = \begin{cases} \frac{n}{2}\xi(x - 1), & y - \frac{1}{n} \leq \xi \leq x \\ \frac{n}{2}x(\xi - 1), & x \leq \xi \leq y + \frac{1}{n} \end{cases} \tag{2}
$$

(b) Show that, for each fixed $y \in (0, 1)$, the sequence of functions $G_n(x, y)$ converges uniformly to a limit which we denote by $G(x, y)$. Show also that $G_n(\cdot, y) \to G(\cdot, y)$ as distributions in χ.

Solution. An obvious proposal for the limit of this sequence is

$$G(x,y)=\begin{cases}(y-1)x & x\leq y\\ (x-1)y & x>y.\end{cases}$$

We seek to bound the uniform norm of $G_n - G$ by a quantity that converges to zero. Observe that $G_n(x,y)=G(x,y)$ in the region $|x-y|>1/n$. Thus,

$$\sup_{x\in[0,1]}|G_n(x,y)-G(x,y)|=\sup_{|x-y|\leq 1/n}|G_n(x,y)-G(x,y)|=(*).$$

Both functions are symmetric about $x=y$, so we only need to consider the interval $y-1/n\leq x\leq y$. Note that $G_n(x,y)=G(x,y)$ and $\partial_x G_n(x,y)=\partial_x G(x,y)$ at $x=y-1/n$. The facts that $\partial_x^2 G_n(x,y)>0$ and $\partial_x^2 G(x,y)=0$ on this interval imply that extrema are on the boundary. Hence,

$$(*)=|G_n(y,y)-G(y,y)|=\left|\frac{n}{4}y^2-\frac{n}{2}\left(y-\frac{1}{n}\right)y+\frac{n}{4}\left(y-\frac{1}{n}\right)^2-(y-1)\,y\right|=\frac{1}{4n}\to 0.$$

In other words, given $\epsilon>0$ it is possible to find $N<\infty$ such that $|G_n(x,y)-G(x,y)|<\epsilon$ for all $x\in[0,1]$ and $n\geq N$. To show convergence in distributions, we find for any $\varphi\in\chi$ that

$$\begin{aligned}\left|\int_0^1(G_n(x,y)-G(x,y))\,\varphi(x)\,dx\right| &\leq \int_0^1|G_n(x,y)-G(x,y)||\varphi(x)|\,dx\\ &\leq \sup_{x\in[0,1]}|G_n(x,y)-G(x,y)|\int_0^1|\varphi(x)|\,dx\to 0.\end{aligned}$$

(c) Compute $\partial_x G(x,y)$ and $\partial_x^2 G(x,y)$ in the sense of distributions on χ.

Solution. See part (b).

In what follows, f is a given smooth function that has compact support in $[0,1]$, and $\delta_y\in\chi'$ denotes the distribution $\varphi\mapsto\varphi(y)$ for every $\varphi\in\chi$.

(d) Show that the sequence of distributions T_n given by

$$T_n=-\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)\delta_{\frac{k}{n}}$$

converges in the sense of distributions. Identify the limit distribution T.

Solution. Let $\phi\in\chi$.

$$\begin{aligned}T_n(\phi) &= \int_K -\frac{1}{n}\sum_{k=1}^{n}f(\tfrac{k}{n})\delta_{\frac{k}{n}}\phi(x)dx\\ &= -\frac{1}{n}\sum_{k=1}^{n}f(\tfrac{k}{n})\phi(\tfrac{k}{n})\end{aligned}$$

It is obvious that, as $n \to \infty$, we will be taking an infinite sum over the region $(0,1)$, and multiplying said sum by an infinitely small number. Hence

$$-\frac{1}{n}\sum_{k=1}^{n} f(\tfrac{k}{n})\phi(\tfrac{k}{n}) \to -\int_0^1 f(x)\phi(x)dx$$

and we have that $T_n \to T_f$, where T_f is the right hand side above.

(e) Show that, for each n, there is a function u_n which vanishes at 0 and 1, such that the distributional derivative $u_n'' = T_n$.

Solution. Generally we have that $u'' = f$. If we look for a distributional solution then, for $\phi \in \chi$

$$\begin{aligned} u_n''[\phi] &= -\frac{1}{n}\sum_{k=1}^{n} f(\tfrac{k}{n})\delta_{\frac{k}{n}}\phi(x) \\ &= T(\phi) \end{aligned}$$

(f) Prove the existence of a function u which gives a (distributional) solution to Poissons equation $u'' = f$ with $u(0) = u(1) = 0$.

Solution. Let the solution to Poissons equation be given by

$$u(x) = \int_{-\infty}^{\infty} k(x,\xi)\phi(x)dx,$$

where we define k in Eqn. 2.

4. **Poissons equation in 3 dimensions**

In what follows $x \in \mathbb{R}^3$ and $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$. Recall, that $\mathcal{D}'$ denotes the space of distributions on compactly supported smooth functions.

(a) Show that PV $(1/||x||)$ defined by

$$\mathrm{PV}\left(\frac{1}{||x||}\right)(\varphi) = \lim_{\epsilon\to 0}\int_{||x||>\epsilon}\frac{\varphi(x)}{||x||}dx \tag{3}$$

is a distribution in $\mathcal{D}'(\mathbb{R}^3)$.

Solution. If the limit does indeed exist for all $\varphi \in \mathcal{D}(\mathbb{R}^3)$, it is clear that $\mathrm{PV}(1/\|x\|)$ is linear since the integral operator is linear. To show the existence of the limit, we may write the principal value in sphericalpolar coordinates as

$$\begin{aligned} \mathrm{PV}\left(\frac{1}{\|x\|}\right)(\varphi) &= \lim_{\epsilon\to 0}\int_{\|x\|>\epsilon}\frac{\varphi(x)}{\|x\|}\,dx \\ &= \lim_{\epsilon\to 0}\int_{\epsilon}^{\infty}\int_0^{2\pi}\int_0^{\pi} r\,\sin(\phi)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr \\ &= \int_0^{\infty}\int_0^{2\pi}\int_0^{\pi} r\,\sin(\phi)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr \end{aligned}$$

To show continuity, suppose $\varphi_j \to 0$ in $\mathcal{D}'(\mathbb{R}^3)$. There exists $R < \infty$ (independent of j) such that $\varphi_j(x) = 0$ for all $\|x\| \geq R$. Then,

$$\begin{aligned}
\left|\mathrm{PV}\left(\frac{1}{\|x\|}\right)(\varphi_j)\right| &= \left|\int_0^\infty \int_0^{2\pi} \int_0^\pi r \, \sin(\phi)\, \varphi_j(r,\theta,\phi)\, d\phi\, d\theta\, dr\right| \\
&= \left|\int_0^R \int_0^{2\pi} \int_0^\pi r \, \sin(\phi)\, \varphi_j(r,\theta,\phi)\, d\phi\, d\theta\, dr\right| \\
&\leq \int_0^R \int_0^{2\pi} \int_0^\pi r \, |\varphi_j(r,\theta,\phi)|\, d\phi\, d\theta\, dr \\
&\leq p_{R,0}(\varphi_j) \int_0^R \int_0^{2\pi} \int_0^\pi r \, d\phi\, d\theta\, dr \\
&= \pi R^2 \, p_{R,0}(\varphi_j) \to 0.
\end{aligned}$$

Thus, $\mathrm{PV}(1/\|x\|)$ is a distribution in $\mathcal{D}'(\mathbb{R}^3)$.

(b) Find a_n and b_n such that the function $\Psi_n(r)$ given by

$$\Psi_n(r) = \begin{cases} a_n - b_n r^2 & 0 \leq r \leq n^{-1} \\ \frac{1}{4\pi r} & r > n^{-1} \end{cases}$$

is continuous and differentiable for $r \in (0, \infty)$.

Solution.

$$\begin{aligned}
\Psi_n(\tfrac{1}{n}) &= a_n - \frac{b_n}{n^2} \\
&= \frac{n}{4\pi}, \text{ and now evaluate the derivative} \\
\Psi_n'(\tfrac{1}{n}) &= -\frac{2b_n}{n} \\
&= -\frac{n^2}{4\pi}, \text{ hence} \\
b_n &= \frac{n^3}{8\pi} \\
a_n &= \frac{3n}{8\pi}
\end{aligned}$$

and then

$$\Psi_n(r) = \begin{cases} \frac{3n}{8\pi} - \frac{r^2 n^3}{8\pi} & 0 \leq r \leq n^{-1} \\ \\ \frac{1}{4\pi r} & r > n^{-1} \end{cases}$$

(c) Show that $\Psi_n(||x||) \to (4\pi)^{-1}\mathrm{PV}(1/||x||)$ in the sense of distributions.

Solution. Suppose $\varphi \in \mathcal{D}(\mathbb{R}^3)$. Then,

$$\int_{\mathbb{R}^3} \Psi_n(\|x\|)\,\varphi(x)\,dx = \int_{\mathbb{R}^3} r^2\,\sin(\phi)\,\Psi_n(r)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr$$

$$= \frac{1}{8\pi}\int_{r\leq 1/n} r^2\,\sin(\phi)\,(3n{-}n^3r^2)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr + \frac{1}{4\pi}\int_{r>1/n} r\,\sin(\phi)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr$$

To show convergence, we observe that

$$\left|4\pi\int_{\mathbb{R}^3} \Psi_n(\|x\|)\,\varphi(x)\,dx - \mathrm{PV}\left(\frac{1}{\|x\|}\right)(\varphi)\right|$$

$$= \left|4\pi\int_{\mathbb{R}^3} \Psi_n(\|x\|)\,\varphi(x)\,dx - \int_{\mathbb{R}^3} r\,\sin(\phi)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr\right|$$

$$= \left|\frac{1}{2\pi}\int_{r\leq 1/n} r^2\,\sin(\phi)\,(3n-n^3r^2)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr - \int_{r\leq 1/n} r\,\sin(\phi)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr\right|$$

$$= \left|\frac{1}{2\pi}\int_{r\leq 1/n} \sin(\phi)\,(3nr^2-n^3r^4-r)\,\varphi(r,\theta,\phi)\,d\phi\,d\theta\,dr\right|$$

$$\leq \frac{p_{1/n,0}(\varphi)}{2\pi}\int_{r\leq 1/n} |3nr^2-n^3r^4-r|\,d\phi\,d\theta\,dr \leq \pi\,p_{1/n,0}(\varphi)\int_0^{1/n} |3nr^2-n^3r^4-r|\,dr$$

$$\leq \pi\,p_{1/n,0}(\varphi)\left(\int_0^{1/n} 3nr^2\,dr + \int_0^{1/n} n^3r^4\,dr + \int_0^{1/n} r\,dr\right)$$

$$= \pi\,p_{1/n,0}(\varphi)\left(\frac{1}{n^2}+\frac{1}{5n^2}+\frac{1}{2n^2}\right) \to 0.$$

(d) Compute $\Delta\Psi_n$ and using this, or otherwise, compute the distributional derivative $\Delta\mathrm{PV}(1/||x||)$.

Solution. We may write the Laplacian operator in spherical polar coordinates as

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\sin(\theta)\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2(\theta)}\frac{\partial^2}{\partial^2\phi}.$$

Fortunately, Ψ_n has no dependence on θ or ϕ. This makes it straightforward to calculate

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}(a_n-b_nr^2) = \frac{1}{r^2}\frac{\partial}{\partial r}(-2b_nr^3) = -6b_n$$

and

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}\left(\frac{1}{2\pi r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{-1}{2\pi}\right) = 0.$$

Thus,

$$\Delta\Psi_n(r) = \begin{cases} -6b_n & 0\leq r\leq 1/n \\ 0 & r>1/n. \end{cases}$$

To compute the distributional derivative, we may evaluate the equivalent limit

$$\Delta \mathrm{PV}\left(\frac{1}{\|x\|}\right)(\varphi) = 4\pi \lim_{n\to\infty} \int_{\mathbb{R}^3} \Delta \Psi_n(\|x\|)\, \varphi(x)\, dx.$$

Since the functions $\Delta\Psi_n$ appear to be approximate delta functions, a candidate limit is $-\varphi(0)$. Observe that $\int_{\mathbb{R}^3} \Delta\Psi_n(x)\, dx = -1$. This implies that we may consider the quantity

$$\begin{aligned}
\left|\int_{\mathbb{R}^3} \Delta\Psi_n(\|x\|)\,(\varphi(x)-\varphi(0))\, dx\right| &= \left|\int_{\mathbb{R}^3} r^2 \sin(\phi)\, \Delta\Psi_n(r)\,(\varphi(r,\theta,\psi)-\varphi(0))\, d\phi\, d\theta\, dr\right| \\
&\leq \int_{\mathbb{R}^3} r^2\, \Delta\Psi_n(x)\, |\varphi(r,\theta,\psi)-\varphi(0)|\, d\phi\, d\theta\, dr \\
&= \frac{3n^3}{4\pi} \int_{r\leq 1/n} r^2\, |\varphi(r,\theta,\psi)-\varphi(0)|\, d\phi\, d\theta\, dr = (*).
\end{aligned}$$

Continuity of φ at 0 implies that given $\epsilon > 0$ there exists a δ such that $\|x\| < \delta$ implies $|\varphi(x) - \varphi(0)| < \epsilon$. We may choose n large enough that $1/n < \delta$. Then,

$$(*) \leq \frac{3\epsilon n^3}{4\pi} \int_{r\leq 1/n} r^2\, d\phi\, d\theta\, dr = \frac{\epsilon}{2\pi}.$$

This quantity can be made arbitrarily small by increasing n. Hence,

$$\Delta \mathrm{PV}\left(\frac{1}{\|x\|}\right)(\varphi) = -4\pi\, \varphi(0).$$

(e) ρ is a compactly supported, smooth function defined on $\mathbb{R}^3$. Write down a (distributional solution to Poissons equation $\Delta\Phi = -\rho$.

Solution. To find a distributional solution, we will look for a solution using the results from the previous problem. Hence we have that

$$\begin{aligned}
\Delta\Phi &= -\rho, \text{ which in distribution becomes} \\
\Delta \mathrm{PV}(1/\|x\|)[\phi] &= -\delta(\phi) \\
&= -\int_0^{\pi}\int_0^{2\pi} \phi(0,\theta,\psi)\sin\theta d\theta d\psi \\
&= -\phi(\mathbf{0})\int_0^{\pi}\int_0^{2\pi} \sin\theta d\theta d\psi \\
\Delta \mathrm{PV}(1/\|x\|)[\phi] &= -4\pi\phi(\mathbf{0})
\end{aligned}$$

But we also have that

$$\begin{aligned}
\Delta \mathrm{PV}(1/\|x\|)[\phi] &= \mathrm{PV}(1/\|x\|)[\Delta\phi], \text{ hence} \\
\mathrm{PV}(1/\|x\|)[\Delta\phi] &= -4\pi\phi(\mathbf{0})
\end{aligned}$$

5. **Wave equation in 3 dimensions (optional)**

(a) Show that the distribution $\frac{1}{t}\delta(||x|| - t)$ is a distributional solution of the wave equation $u_{tt} - \Delta u = 0$ in 3 dimensions, for $t > 0$.

Solution. Let $T = \frac{1}{t}\delta(\|x\| - t)$, then we have

$$\begin{aligned}
&T_{tt} - \Delta T = 0 \\
T_{tt}(\phi(r,\theta,\psi)) &= T(\phi_{tt}(r,\theta,\psi)) \\
&= \tfrac{1}{t}\delta(\|x\| - t)\phi_{tt}(r,\theta,\psi) \\
&= \frac{\phi_{tt}(t,\theta,\psi)}{t}, \text{ and} \\
\\
\Delta T(\phi) &= T(\Delta\phi) \\
&= \tfrac{1}{t}\delta(\|x\| - t)\Delta\phi(r,\theta,\psi) \\
&= \frac{\Delta\phi(t,\theta,\psi)}{t} \\
&\Downarrow \\
\phi_{tt}(t,\theta,\psi) &= \Delta\phi(t,\theta,\psi)
\end{aligned}$$

As we desired.

(b) Does $\lim_{t\to 0^+} \frac{1}{t}\delta(||x|| - t)$ define a distribution $\mathcal{D}'(\mathbb{R}^3)$?

Solution. Time ran out.